Resolving Resolution Uncertainty

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Pantex Metrology

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Topics

The Issue

UUT uncertainty sources–To budget or not to budget?

- Brewing controversy at the water cooler
  - Traditional: Include anything that obscures the UUT error.
  - Recent view: UUT errors belong to the UUT, not the lab.
- Both viewpoints have persuasive elements.
- Real issue–high costs

Full Disclosure

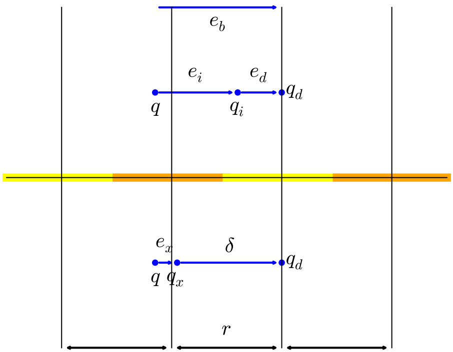
- Neutrality, objectivity, measurement science
- Biased presenter
  - TAR $= T_{uut}/T_{std} = apples/apples$
  - *Z540.3* TUR problems
    - TUR $= T/U = apples/oranges$
    - Removed our 4:1 crutch?
    - UUT resolution & repeatability may drive TUR below 4:1.
  - TUR gauges measurement decision risk–the goal
  - Invented new crutches
  - UUT resolution: Too bad, get over it.

Learning Objectives

1. Understand measurement process error sources and when they apply.
2. Understand the consequences of UUT error contributions.
3. Learn measurement and analytical methods to mitigate UUT resolution.

Error Model

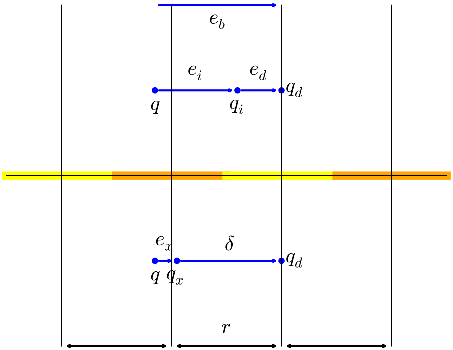
Back to fundamentals



- UUT process
  - Stimulus q
  - Internal error e_i
  - Internal quantity q_i
  - Rounding or display error e_d
  - Indication q_d
  - Overall bias e_b
- Calibration reference
  - Stimulus q
  - Excl. process error e_x
  - Reference quantity q_x
  - Apparent error δ

Error Model

Back to fundamentals

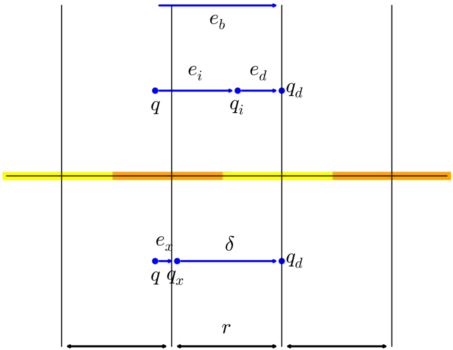


Traditional Analysis

- Determine e_i
- Rounding obscures e_i
- So $u = \sqrt{u_{\text{res}}^2 + u_x^2}$
- Why e_i? Justified?

Error Model

Back to fundamentals



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Recent Proposals

- Determine e_b
- e_i, e_d irrelevant
- GUM: no gross errors
- No uncertainty in q_d
- Therefore $u = u_x$

Error Model

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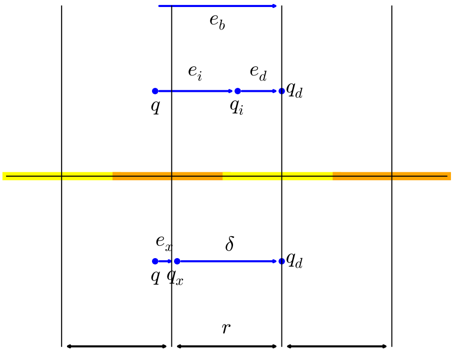

Zero PFA and PFR on q_d
Problem over?

Traditional Analysis

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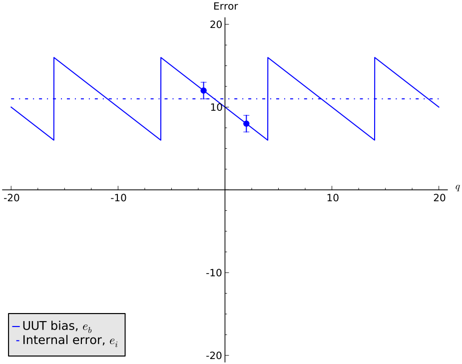
Assumptions

- The "obviously" trap: perfect GD&T drawings, schematics, mathematics
- Traditional analysis: assumes e_i, e_d matter
- New proposal: e_i, e_d do not matter
- What do we want? (Why do we calibrate?)
- This talk's facilitating assumptions:
  - A perfect measurement process other than the UUT: $u_x = 0$
  - Focus on resolution–no other UUT errors
  - e_i constant over the measuring interval and over time

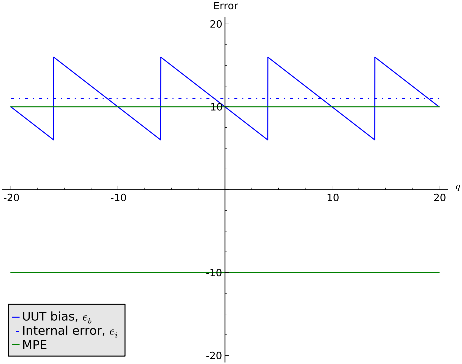
Measurement Equivalence



Lab failure: Two e_d measurements may not agree–no equivalence.

1. Ignoring u_{res} fails for VIM-defined calibration by lack of agreement – the point of metrology.
2. Calibration by this definition requires e_i, e_d characterization (values and uncertainty).

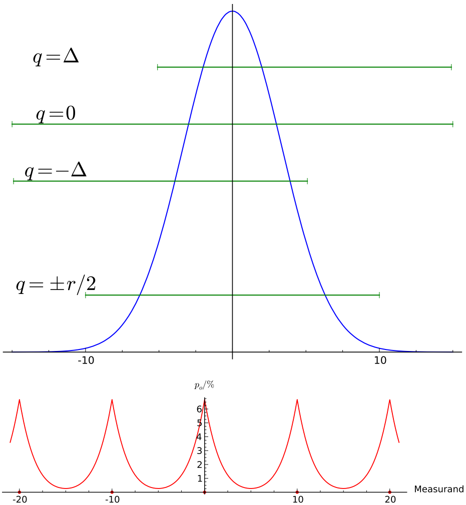
Conformance Test



The conformance test may pass but leave many usage points outside MPE.

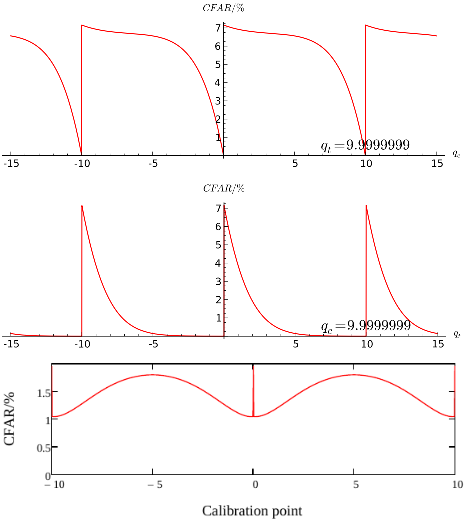
1. Lab has zero PFA and PFR
2. Not so for customer measurements.
3. Another assumption: Customers want accuracy over their measuring intervals.
4. Ignoring u_{res} fails for conformance testing also.
5. RP-12: e_i, e_d matter.

Internal Error Acceptance Bounds



- MPE $= r = 10$
- $\Delta = r/100$
- $u_i =$ MPE$/3$
- Different calibration points restrict the internal error by different amounts.
- A given cal point may not sufficiently restrict measurement error across the nominal indication.

Unmitigated Measurement Decision Risk



- Calibrating at a usage point helps little elsewhere.
- Using the UUT away from the calibration point invites trouble.
- Calibrating just off an indication value minimizes customer's expected CFAR.

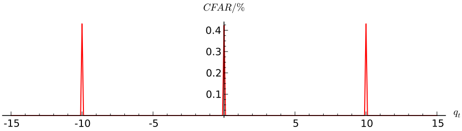
Value Proposition

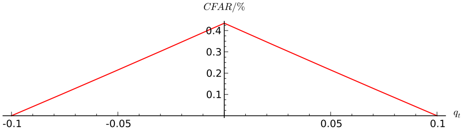
Theorem

During calibration, all measurement errors belong to the lab.

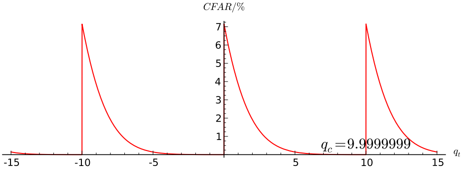
- Many ways to acknowlege UUT uncertainty and still perform a quality measurement
- Value decision: Avoid undue costs
  - Does the application justify the tight specification and expense?
  - Yes: Say what you do, do what you say.
    - Redesign the instrument (more digits, signal output).
    - Beef up the calibration process.
  - No: Change the spec!
    - Internal or external customers
    - RP-1 Survey: 2/3 of NCSLI members calibrate to other specs

Nominal Value Sandwich





Compared to original



- Calibrate above and below nominal
- Same or different points
- Excellent risk reduction
- Measure close to nominal but safely away for process uncertainty.
- It probably happens by chance already.

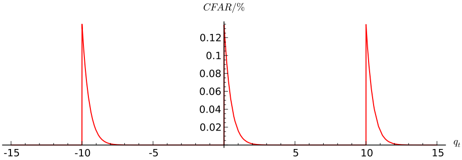
Dither, Slew

- Dithering
  - Old technique
  - Multiple measurements within the resolution interval
  - May occur in different intervals
  - Two or three samples eliminates resolution conditions.
- Slewing
  - Mimic the analog meter technique
  - Assumes a finely variable reference
  - Determine the switching point
  - One point: $e_i = \frac{1}{2}\,\text{LSD} + q_n - q_x$
  - Two points: $e_i = q_n - \frac{1}{2}\,q_{p_1} - \frac{1}{2}\,q_{p_2}$
  - Eliminates UUT resolution down to adjustability
- Expensive–automate

Guardbands

- Implicit Guardband
  - For specs with a component proportional to stimulus
  - Example: $\pm(1\,\text{mV} + 1\,\% \cdot s)$
  - $s = 16\,\text{mV}$, $\text{MPE} = 1.16\,\text{mV}$
  - RSS guardband acceptance limit: $A = 1.0061\,\text{mV} > 1\,\text{mV}$
- Full Guardbands
  - Rounding errors cancel $e_i < \frac{1}{2}\,r$.
  - Assumes a nominal reference value
  - Accept only exact indications
  - For instruments with a claim to their MPE
  - Significantly reduces CFAR

Risk Analysis



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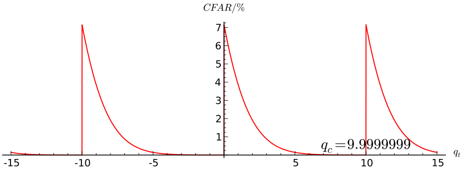

- *Z540.3* 2 % FAR option?
- Reasonably estimate the internal error distribution
- Historical measurement reliability, acceptance bounds
- Perhaps $u_i = \frac{1}{2} r/3$.

Test Point Selection

- The real world worsens the problem.
  - The measurement process adds uncertainty.
  - Internal error may drift after calibration.
  - Internal error may vary over the UUT measuring interval.
- Range calibration principles apply.
  - Choose points appropriately to sample the instrument state.
  - Sample the quantization also, if significant.
  - The selection should account for all UUT effects.
  - Point selection should ensure the user's measurement quality.
  - Joe Petersen, "Principles for Calibration Point Selection", *Measure*

Conclusions

- Improving the metrology works; ignoring UUT errors does not.
- A measurand defined as an uncertainty-free UUT indication does not
    - Adequately calibrate the instrument,
    - Address performance over a measuring interval,
    - Conformance test an instrument without undue risk.
- Real solutions exist to reduce and account for UUT errors.
- During calibration, all errors belong to the calibration laboratory and flow downstream to the user.

A Resolution

- Metrology searches for truth and should recognize reality.
- Know why you include uncertainties in the budget.
- Reduce errors where practical and quantify their remaining uncertainty.
- Include all uncertainties the calibration process has not thoroughly marginalized.
- Caveat Emptor–Take care when
  - Purchasing calibration services that do not address UUT errors,
  - Following standards that recommend such practices.

Questions

Plenty to think about for all parties: users, metrologists, manufacturers, accreditors, standards committees

Open to panel discussions, meetings, off-line discussion

Full paper with slides available for review soon

Acknowledgments

- Pantex Metrology
- Cherine-Marie Kuster

Thank You for your time! Questions?

Resolving Resolution Uncertainty

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Abstract

ANSI/NCSLI Z540.3 redefined the "test uncertainty ratio" (TUR) measurement quality metric as a ratio of maximum permissible error to measurement process uncertainty. The definition engendered some immediate and lingering controversy. Much of this disagreement stems from inconvenience: Including uncertainty components from the unit under test (UUT) clouds conformity assessments when an instrument's maximum permissible error (MPE) specification falls near its repeatability or resolution limits. The TUR then apparently loses its value as a simple measurement process validation. Some practitioners view UUT error contributions a simple reality to address in calibration processes, others consider such inclusions, at least in some circumstances, a logical fallacy. Unfortunately, differing opinions also arise between laboratory assessors and, by implication, between accreditation bodies. This paper neutrally reviews the problem from first principles and evaluates the two positions in terms of metrological compatibility and measurement decision risk. The paper also discusses mitigation methods.

Learning Objectives

1. Consider measurement process error sources and when they apply.
2. Consider the consequences of UUT error contributions.
3. Name measurement and analytical methods to mitigate UUT resolution.

Background

If measurement science conferences had the fabled water cooler where all the important conversations happened, listening to recordings from a hidden microphone would reveal a growing and controversial dichotomy in the analytical metrology world: to include or not to include UUT resolution (and other UUT error sources) in a measurement uncertainty budget. Convention attendees may well have heard adherents from one or both viewpoints vigorously express their opinions, perhaps convincingly, leaving them uncertain as to the correct choice.

Lying near the heart of the matter, the currently most ubiquitous TUR[1] definition [1] prescribes a minimum 4:1 ratio of MPE to expanded measurement process uncertainty. Many prior TAR[2] and TUR interpretations judged calibration process adequacy by simple apples-to-apples[3] ratios of UUT-to-measurement-standard tolerance intervals or uncertainties. The

[1]Test uncertainty ratio
[2]Test accuracy ratio
[3]Bodark apple to Washington Fancy in some cases, since no one really knew the associated confidence levels

Z540.3 definition, however, lumps all process uncertainties in the denominator, not just that of the measurement standards. Since the UUT typically contributes error to the measurement process, the UUT uncertainty components suddenly appeared in both the ratio's numerator and denominator. Thus, when the UUT sources dominate, the TUR metric ironically indicates an inadequate process through no apparent fault of the measurement method or standards. In the case at hand, a ± 1 LSD[4] MPE yields a ≈ 0.29 LSD UUT resolution uncertainty and the dreaded $\approx 1.7{:}1$ TUR.

Left at that, the situation would impact measurements and incur appreciable costs. NASA, for example, identified [2] uncertainty analysis as the largest *Z540.3* implementation cost. NASA bears the full analysis expense for those calibrations lacking evidence for EOPR[5] $\geq 89\,\%$ or TUR $\geq 4.6{:}1$. Manufacturers who produce instruments that tightly control measurement error wish to specify and validate resolution-level MPE compliance in a way that meets quality requirements; their customers have a similar stake. Laboratories who now question the correct approach wonder what position their next assessor will take, as indeed we find assessors with opposing viewpoints [3, 4]. By way of full disclosure, this author initially resisted the *Z540.3* TUR definition for costs reasons that included the UUT uncertainty issue, but later acceded and then defended and did not question the stance again until researching this paper.

We will attempt to neutrally and objectively discover underlying reasons for the disagreement and thereby reconcile the two apparently opposing arguments and resolve the issue. After all, given a problem in pure metrology, the conflict should give way to logic, mathematics and measurement science. We lay out an intuitive logical argument based largely on error analysis and the resulting measurement decision risk. To unclutter the flow, we postpone the derivations and other details to an appendix. We also provide mitigation methods to reduce expense. To ground itself more rigorously, the paper's language follows the *VIM* [5], *RP-18* [6], and *JCGM 106* [7] where applicable.

1. Scope

Let us first focus our investigation. Whether attributing it to the display itself or the operator who reads it, all parties seem to agree that an analog indication, if not at a scale mark, adds a resolution component to the uncertainty budget, so this paper centers on digital indications. Furthermore, 3 LSD MPE specifications provide adequate ≥ 5.2 TURs by the standard calculation and thus sidestep the controversy for those living by the TUR. With negligible non-UUT uncertainty components, one may guardband a 2 LSD MPE to obtain acceptance limits no less than one LSD, marginalizing the issue.

Therefore, without loss of generality, we primarily concentrate on the game-changing resolution conditions, 1 LSD MPEs, where we no longer have the luxury of tossing in a generic and negligible uncertainty component and ignoring the measurement impacts. Note that digital resolution error may occur not only due to limited UUT display capability, but also due to the measurement standard indication[6] or rounding the measurement result upon recording

[4] Least significant digits, or counts

[5] End-of-period reliability

[6] Typically subsumed in the measurement standard's uncertainty statement

it. All three errors may occur in the same measurement process, but the UUT error contributions dominate our interest so we assume negligible measurement standard uncertainty and artificial rounding unless otherwise noted.

2. The Measurement Error Model

We begin with a neutral measurement process definition. Figure 1 depicts a subset of the UUT's discrete display values with spacing equal to the UUT resolution, r. The UUT measures the unknown measurand, q, with some unknown internal error, e_i, to arrive at its internal estimate, q_i. Depending on the UUT at hand and its measurement principle, e_i likely comprises a number of error sources, but they do not concern us individually. The UUT does not provide q_i externally, but instead adds a rounding, or display error, e_d, to display its measurement result, q_d. The UUT will indicate q_d for any q_i falling between the limits $q_d \pm \frac{1}{2}\,r$, as Fig. 1's colored bars show.

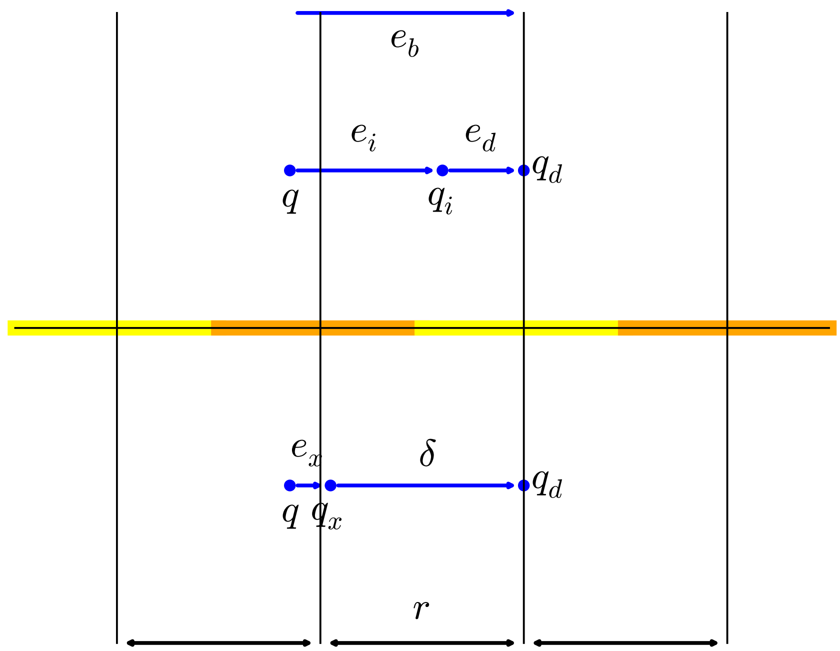

Figure 1. Measurement error model depicted among display values (vertical lines).

To ensure UUT measurement quality, we relate the UUT indication to traceable quantity values via measurement standards. Figure 1's lower section shows this: A traceable measurement process realizes q plus an unknown process error, e_x, as its reference value, q_x. For a given measurement method, we define e_x to include non-UUT errors stemming from measurement standards, environmental factors, operators, repeatability, rounding, computation, etc.; again, the individual components do not matter here. This model thus separates the UUT, e_b, and measurement process, e_x, errors to help clarify the issues. From Fig. 1, we see the traceable relation $e_b = \delta + e_x$.

A pure conformity assessment stops there and simply seeks to determine with high confidence whether or not the UUT's measurement bias $e_b = q_d - q$ lies between the MPEs. In other words, if the UUT passes, the UUT user may take $q \approx q_d$, understanding that $|\delta| \leq$ MPE. Calibration, though, has a second step [5] that establishes a corrective relation for obtaining a measurement result from the UUT indication [5]. In our case, the lab would report a post-calibration relation $q = -e_b + q_d$, with which the UUT user would remove e_b from the indication.

High-level quality standards [1, 8] require that conformity assessments account for, and that calibrations report, the measurement uncertainty. Anything in the measurement procedure that obscures the measurand contributes to its uncertainty. The critical question becomes: What measurand, and thus what uncertainty, do we want? As we see in the next sections, different answers lead to different results.

3. Established Practice, New Ideas

Most of the literature and published analyses before about 2010 seem to assume that UUT uncertainty sources go into the budget. Some, as we shall see, explain their viewpoint, but not many. Practitioners involved with electrical-parameter measuring instruments did not seem to stumble over resolution uncertainty too badly. Most of those instruments relegate resolution problems to the seldom-used low ends of their measuring intervals and even there, other measurement uncertainties often subsume the resolution.

Though the industry may have accepted the established practice for electrical quantity measurements, not so in the dimensional measurement world. Here manufacturers routinely specify instruments such as the infamous micrometer to ±1 LSD over a wide measuring interval. Anyone who then included UUT uncertainty sources in the budget while attempting to adhere to dimensional acceptance standards such as *ISO 14253-1:1998* [9], revised 2013, and *ASME B89.7.3.1-2001* (R2011) [10] drew a "Do not pass GO" card. Alternative approaches began to appear in 2010 [4, 11, 12, 13] and their proponents began revising applicable standards, e.g., *ASME B89.1.13-2013* [14] with the philosophy of excluding UUT uncertainty.

The papers [4, 13] recently began leaning on the *VIM* and defining the measurand as the UUT indication error (e_b in our model) as a key to this philosophy. Quantifying the logic with our model, we have the measurand $e_b = q_d - q$ and the measurement result $\delta = q_d - q_x$. The GUM [15] excludes gross errors such as transcription mistakes from uncertainty analysis, so recording a digital indication to its full resolution incurs no uncertainty. We therefore attribute no uncertainty to q_d and it follows that for any given measurement, the UUT bias uncertainty, u_b, simply equals the measurement process uncertainty, u_x. No UUT uncertainty sources involved.

This seems to make intuitive sense. At the calibration point, the measurand includes all effects from UUT resolution, repeatability, etc. and the new idea therefore seems to have accomplished its goal to ensure UUT measurement quality without incurring an uncertainty penalty. Problem over? Let's see.

4. Enter Measurement Decision Risk

Measurement uncertainty, TURs, or arbitrary decision rules only indicate conformity assessment quality indirectly; MDR[7], the risk of making the wrong acceptance decision draws a more solid bottom line. If we look at the UUT indication's FA[8] and FR[9] conditions (see the appendix), we find that a perfect measurement will never cause an FA or FR on the UUT indication; the logic still holds, the point seems proven.

As it turns out [16], we know no system of absolute proof, not even for simple arithmetic or logic rules. Ultimately, every conclusion rests on how well it matches our observations, and in turn on how well our assumptions fit the purpose. Many arguments arise from unstated assumptions, including ours. This case has at least two implicit assumptions:

- The calculation δ has no uncertainty.
- We have chosen the correct measurand.

We may satisfy the first assumption by imposing the restriction that no rounding take place on δ (ignore significant figures rules and do not match the indication's digit count) before the conformity assessment or in reporting it; otherwise, we reintroduce the very resolution uncertainty value we wish to avoid. For example, rounding a 0.0013 mm measurement result to 0.001 mm and accepting the item to a ±0.001 mm MPE constitutes a false accept; the rounding would require a guardband that accounts for the artificial resolution uncertainty.

What of the second assumption? If indication error represents the appropriate measurand, why does established practice include UUT uncertainties? Surely the *Z540.3* authors didn't put UUT factors in the TUR denominator by accident. *NCSLI RP-12* [17] and previous papers, e.g., [18], clarify what most literature leaves unstated: Previous practice assumes a different measurand: $e_i = \delta - e_d + e_x$; in other words, that we wish to characterize the UUT's internal error before rounding. Its uncertainty, u_i, then appears to involve the rounding uncertainty u_d. Voila! Now we may resolve the issue by determining the most appropriate measurand.

Why would the hidden internal error concern us? Because the UUT's internal error will govern its behavior throughout its measuring interval. If the UUT user may measure any unknown value as opposed to just those values we happened to calibrate or conformity assess, then the internal error means everything. If we wish to know the instrument's state over its entire measuring interval without taking an infinite number of measurements at calibration time, then we certainly want to estimate or bound its internal error and its variation pattern over the interval.

Using the internal error to evaluate false accepts, the conditions become (see the appendix) $e_b \leq \text{MPE} < e_i$. In other words, if we accept the UUT indication at face value at the calibration point when the internal error exceeds the MPE but q_i rounds toward the nominal value, we will have falsely accepted the internal measurand, which may cause a false accept on the external measurand at some other point in the measuring interval. This does not

[7] Measurement decision risk

[8] False accept

[9] False reject

change the zero false accept probability on the external measurand at the point tested, but it does raise the possibility that untested external measurands will exceed the MPE. Ignoring the internal measurand, however, does not cause false rejects, because a true reject anywhere means a true reject overall.

To begin illustrating the problem, Fig. 2 shows a UUT bias resulting from a constant internal error, $e_i = 11$, and MPE $= r = 10$. The display error cycles up and down along the measuring interval due to rounding, sometimes countering, sometimes augmenting the internal error. Added to the internal error, e_d causes the bias to repeatedly cross the MPE, obscuring the internal error and confounding the external error. Thus, a given calibration point selection may happen to find the external measurand error conforming to the MPE at one point but miss non-conforming error at another.

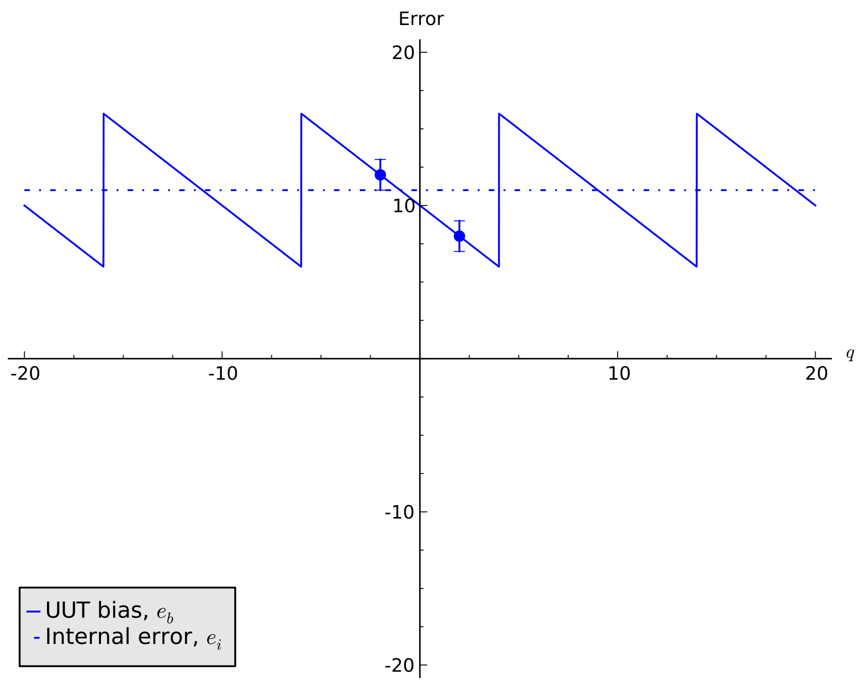

Figure 2. Example UUT bias as a function of the calibration point.

Figure 3 exemplifies the indication's non-conformance probability over a measuring interval if we assume a normally distributed internal error (Eq. 12) with uncertainty MPE/3.[10] The probability varies depending on whether the internal value will round toward or away from the nominal indication and where the measured value falls relative to the discrete indications to either side. For example, just above a nominal value, we accept the external measurand e_b if $-\frac{1}{2}\,r \leq q_i - q_n < \frac{3}{2}\,r$, with q_n the nominal value. That interval, however, allows too much positive error for a measurand just below a nominal value, inviting the false accept. The non-conformance probability hits a minimum, $\approx 0.00066757\%$ in this case, if we measure

[10] An arbitrary choice but less fanciful than the prevailing practice that assumes manufacturer specifications equate to 95 % coverage intervals. The probabilities stated herein all depend on this assumption but their absolute values do not matter. We present them only as comparisons between the various scenarios considered.

something exactly at the discrete indication values but we have zero probability that an unknown q falls exactly there. Maximum vulnerability occurs just off an indication value.

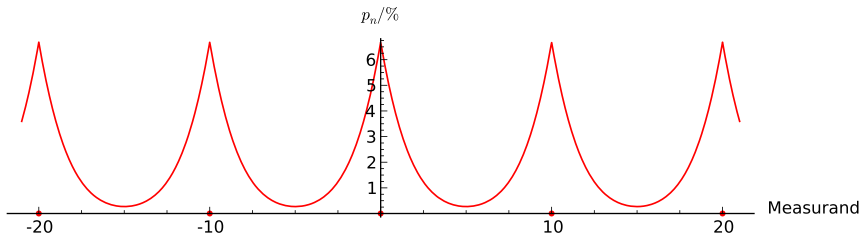

Figure 3. Example non-conformance probability.

To discover the optimum calibration point to cover that vulnerability, we plot in Fig. 4 the probability (Eq. 14) that the user's indication error for a worst-case unknown value q_t falling just below an indication value will exceed the MPE, given that we accepted the instrument at the calibration point, q_c. We call this risk CFAR[11].

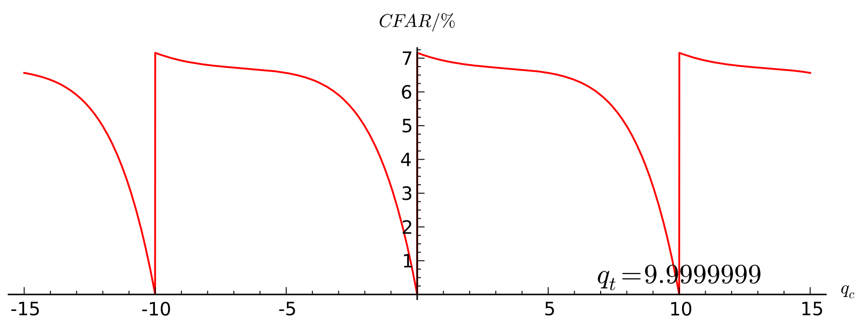

Figure 4. User's worst case CFAR as a function of the calibration point.

Clearly, the calibration point choice critically affects the user's measurement reliability, and apparently, one calibration point per indication does not suffice! In fact, shifting q_c effects the results more than changing its uncertainty. Calibrating by the external measurand ensures reliability at the test point and nowhere else. We should note that we have also assumed $e_x = 0$ and a constant, repeatable, internal error over the UUT measuring interval with no drift after calibration–the situation only worsens in the real world. Figure 5, also from Eq. 14, shows the user's risk at various usage points for an apparently optimum q_c based on Figs. 3 and 4.

To understand the user's overall risk (Eq. 15), Fig, 6 plots that vulnerability versus the calibration point when measuring random values and Fig. 7 how quickly that risk decreases for a near optimum $q_c = \left(\frac{1}{10}\right)$ as the MPE specification increases to ± 2 LSD. The user's risk will not decrease significantly if we select multiple calibration points across the range unless we take care that they vary significantly in relation to their nominal values. Based on these results, we find the practice of accepting UUTs at face value per the external indication unacceptable without an MDR evaluation.

[11]Conditional false accept risk–the customer's risk. The lab's unconditional false accept risk (UFAR) runs slightly lower [6].

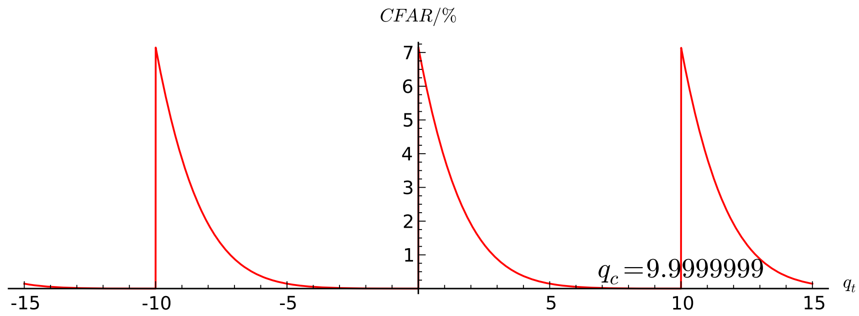

Figure 5. User's best case CFAR as a function of the measured value.

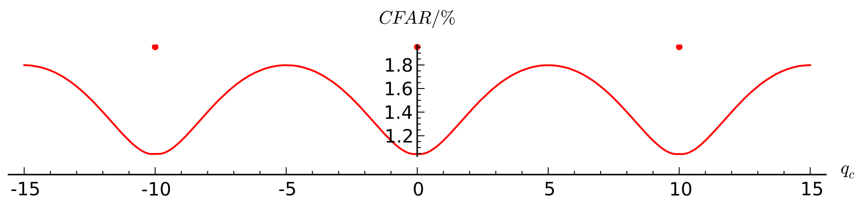

Figure 6. User's average CFAR as a function of the calibration point.

5. Metrological Compatibility

We see immediately from Fig. 2's uncertainty bars that ignoring UUT resolution uncertainty also fails in terms of degree of equivalence, or metrological compatibility. Two laboratories or even the same laboratory with two different standards may determine two completely different UUT biases (whose uncertainties do not overlap), depending on the chosen reference values within the resolution interval. A PT[12] or ILC[13] result then would rely on how closely two laboratories select reference values, rather on how well they perform the measurements. We might argue such a UUT does not merit PT-ILC use. Consider though that every displaying instrument we might use in PTs or ILCs has limited resolution. So, shall we blame the lack of metrological compatibility on a poor instrument choice, a poor measurement process, or a poor uncertainty analysis?

Clearly, laboratories should not claim uncertainties their measurement processes do not support, which means acknowledging all inherent error sources. If we include UUT resolution uncertainty, then two measurements chosen randomly in the resolution interval will agree with probability equal to their coverage interval's confidence level.[14] Likewise, if we alter the measurement process to sufficiently characterize the rounding operation, then the two measurements will agree without including UUT resolution uncertainty. But if we do neither, we lack compatibility, regardless of the instrument at hand. Given that metrology seeks metrological compatibility above all else, the recent proposals fail for both conformity assessment and VIM-defined calibration. In the next section, however, we consider mitigations.

[12] Proficiency test

[13] Inter-laboratory comparison

[14] Assuming we calculate uncertainty via [19] since we do not have a normal distribution.

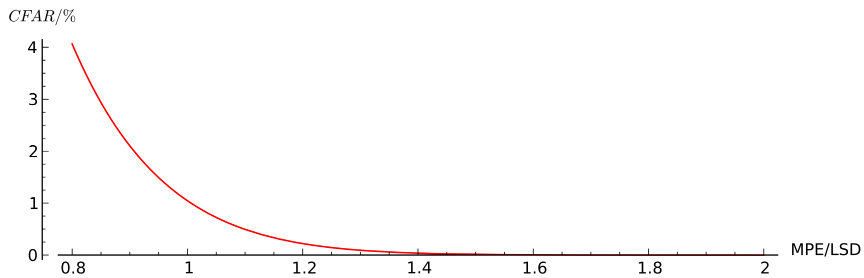

Figure 7. User's average CFAR dependence on MPE.

6. Mitigations

If a candidate measurement process includes an error source that unacceptably degrades the measurement uncertainty, we replace the process with an alternative that avoids, corrects, or suppresses the undesired error, UUT rounding in this case. All errors the lab's calibration process does not address, UUT and otherwise, belong to the calibration laboratory; UUT errors do not get a "Pass GO free" card. This section discusses techniques that both the calibration laboratory and the instrument user may apply to reduce resolution error impacts. We divide the mitigations into process improvements and administrative solutions.

6.1. Process Improvements

First, we look for solutions embedded in the instrument, then we seek measurement method changes to remove the problem UUT error source from the process.

6.1.1. Pre-Digitization Indications

Section 2 postulated that the UUT does not provide q_i externally. Obviously, for instruments that do output a non-visual indication, we may avoid the resolution issue altogether. For example, some air flow meters indicate the air flow value on a limited-resolution display but also output a voltage indicating the flow before digitization. We may then calibrate and correlate the voltage values to flow values and use a voltmeter with the desired resolution.

6.1.2. Nominal Value Sandwich

A conforming measured value at a given calibration point restricts the internal value to varying degrees below and above the point. A calibration point just below a nominal value constrains positive errors within an interval $\approx r$ shorter than that for negative errors and vice versa for a point above the nominal value. If we sandwich the nominal value between two calibration points, taking care to place them far enough from the nominal value to account for their uncertainty u_x, we may drastically reduce the user's risk by minimizing the entire acceptance interval per Eq. 16. CFAR goes to zero except inside the sandwich. Under resolution conditions, point placement effectively takes the role that guardbanding would otherwise play.

Compare Fig. 8 (expanded view Fig. 9) and Fig. 10 (points placed $0.01r$ from the nominal value) to Figs. 5 and 7, respectively. One might calculate an optimum point placement by balancing the risk due to proximity against that due to measurement process uncertainty (not shown). Measurements employing material measures don't allow easy point selection but we may select the closest available values. One point pair covers point placement risk

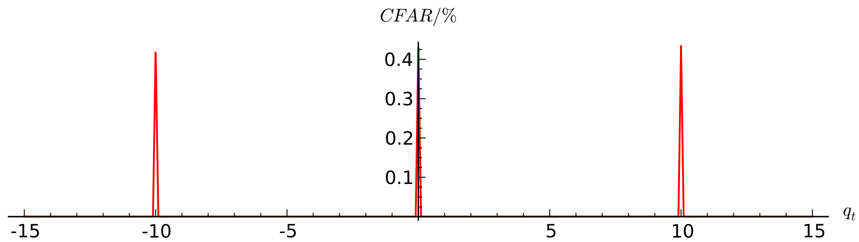

Figure 8. Sandwich CFAR as a function of the measured value.

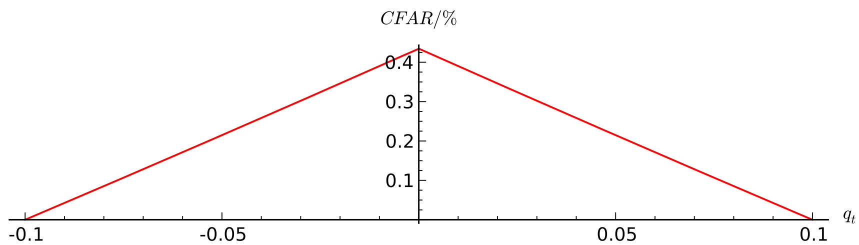

Figure 9. Sandwich CFAR near the nominal values.

throughout the measuring interval; pairs spread over the interval guard against varying internal error as usual with range calibration. The sandwich negates resolution to the extent that the constraint intervals do not exceed $\pm\frac{1}{2}r$; the analysis should account for the remaining resolution uncertainty.

6.1.3. Dithering

The sandwich solution represents a form of dithering, long known to improve measurement resolution. In fact it resembles an old technique that used staggered sensors with optical encoders to subdivide the fixed scale. A typical dither adds n random or preselected values to the signal of interest, averages the result and thereby reduces the resolution to r/n.

If we do not have appropriate sandwich standards, we may theoretically dither our measurement by measuring multiple points evenly spaced over a resolution interval. In practice we probably do not have such finely spaced material measures. The collection of standards used over the measuring interval, however, amounts to the same thing, if we take advantage of their varying differences from the nominal indications. For example, a standard that hap-

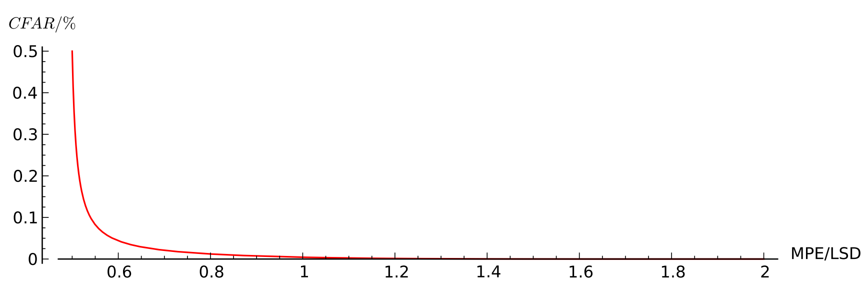

Figure 10. Average sandwich CFAR dependence on MPE.

pens to fall near mid-resolution reduces the resolution uncertainty to $\approx n/2$. Alternatively, we may group the collection into sandwiches and take credit for their spacing above and below nominal indications. Past calibrations employing multiple points over the measuring interval may well have inadvertently reduced CFAR for the user by this effect.

6.1.4. Slewing

When an analog display's resolution significantly affects a conformity assessment, we eliminate the error when practical by adjusting the measured value so that the UUT display indicator lines up with a scale division mark. The standard indication's difference from nominal then gives us the UUT's (negated) error without its resolution error. A digital display does not show us the "needle" moving between graduations but it does, however, plainly indicate when a digit changes. Therefore, we may slew the measurand to find the point where the UUT measurement result increases one digit from the nominal, q_n. Alternatively, we may slew plus and minus one LSD from nominal. In this case, we have the extra advantages of dividing the repeatability by $\sqrt{2}$ and reducing any hysteresis error. Either way, we remove the UUT resolution uncertainty from the picture, solving our problem. See the appendix for detail.

Slewing estimates the internal error value rather than simply bounding it and also serves to derive a calibration correction without resolution uncertainty. CFAR then reverts to the standard formulations [6, 7] based on u_x. Of course, only measurement processes with sufficiently controlled and variable standard indications allow the slewing option (for both analog and digital displays). Also, this technique assumes that the internal UUT error remains constant over small measurand changes.

6.2. Administrative Solutions

Administrative solutions bound the risk by alternative analysis or altering the acceptance interval. First of all, lawyering the so called expanded uncertainty, U_{95}, in the TUR rule doesn't work: We might argue for $k_{95} \approx 1.96$ instead of the arbitrary $k_{95} \equiv 2$, but that only gets us to TUR ≈ 1.8. We might also note that the standard GUM [15] procedure, by its own admission, applies weakly due to the predominant non-normal error distribution. If the rectangular resolution error distribution marginalizes all other components, then $U_{95} \approx 0.95r/2$ and TUR ≈ 2.1. That offers more room for guardbanding but doesn't eliminate the issue outright. So much for lawyers.

6.2.1. Risk Analysis

Typically, quality systems offer TURs as a shortcut to uncertainty analysis, not a mandate. Falling back to uncertainty and risk analysis may alleviate the issue. Though more difficult to calculate without appropriate software, the *Z540.3* maximum 2 % UFAR rule typically relaxes the restraints relative to a 4:1 TUR.

For example, if we assume the same e_i distribution as in our risk examples, a $u_x = \frac{1}{8}r$ measurement process uncertainty for a 4:1 TUR without u_{res}, but then include u_{res}, we have $u_p = \frac{1}{24}\sqrt{3}\sqrt{19}r \approx 0.31r$ (Eq. 2), TUR $= \frac{4}{19}\sqrt{3}\sqrt{19} \approx 1.6$ (Eq. 3), but $CFAR \approx 0.108\,\%$ ([6] with limits per Eq. 10) for the worst case calibration point, well below the 2 % limit. That technically satisfies *Z540.3*, since UFAR falls even lower [6]. It will not please the customer, however, since that analysis alleviated the user's CFAR at uncalibrated points not a bit.

A risk analysis that respects our obligation to the customer should therefore start by estimating the key factor, u_i. We might use manufacturer specification details, historical measurement reliability, or other sources. If we find $u_i = \frac{1}{6}r$ for instance, then comparing Fig. 11 to Fig. 5 shows that deriving a respectable estimate for u_i and applying *Z540.3* to the user's CFAR does the trick.

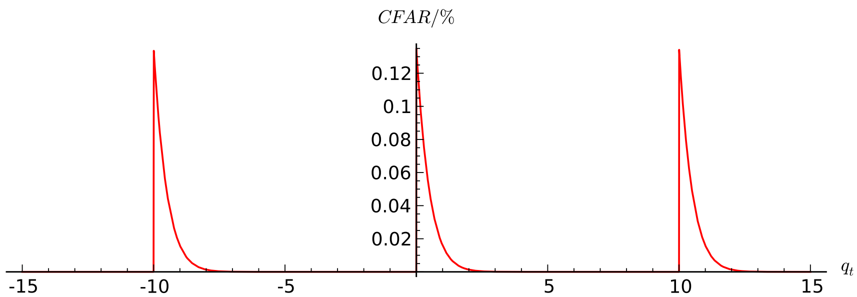

Figure 11. CFAR as a function of the measured value for an example risk analysis.

Note that u_i has complexities if we estimate it from UUT indications with no strategy. The UUT rounding error varies directly with the unknown internal value $q_i = q + e_i$, sometimes canceling e_i partially, sometimes augmenting it, but always decreasing as e_i increases between the rounding points. The two therefore anti-correlate perfectly over a resolution interval but have near zero correlation over a large measuring interval due to e_d's periodicity. So, for example, if estimating u_i from measurement reliability, we should take care to use Eq. (10) for the error limits and not simply the MPE.

6.2.2. Implicit Guardbands

Under some resolution conditions, such as calibrating the low end of an measuring interval belonging to an instrument with a specification $\pm(\mathrm{MPE} = nr + s_o)$, the other specification part gives some leeway beyond the resolution-limited value. Recording the external measurand to the UUT resolution in effect changes the downward rounded MPE limits into guardbanded acceptance limits on the actual MPE. An RSS[15]-guardband or *Z540.3*-based UFAR guardband may push the acceptance limits no tighter.

For example, if the MPE came to 0.0014 unit with resolution 0.001, only the indication errors -0.001 unit, 0.000 unit, and 0.001 unit would pass. If the calculated acceptance limit falls above 0.001, we have an implicit guardband that solves the issue. See the appendix for derivations of the MPE margins required for different guardband options. Ignoring process uncertainty in the case at hand, the margins come to

RSS	Eq. (20)	$s_o \approx 0.00015$
80 %	Eq. (21)	$s_o \approx 0.00025$
G8	Eq. (23)	$s_o \approx 0.00058$

for a few guardbanding options. We recommend evaluating a given implicit guardband situation with a customer CFAR calculation.

[15] Root-sum-square, also known as RDS (root difference of squares)

6.2.3. Full Guardband

If the UUT truly has a claim to its tight MPE, it likely will indicate with zero error for a reference value sufficiently close to the nominal value. The internal rounding cancels absolute internal errors less than $\frac{1}{2}r$ subject to offsets from the measurement standard. The rounding zone allows us to require the UUT to indicate the rounded measurand exactly, without rejecting instruments. Figures 12, 13, 14, and 15 use Eq. 24 in the CFAR equations to depict comparison results for our running example. A full guardband reduces the user's CFAR to half that of the nominal value sandwich. Figure 14 tells us to select calibration points close to the nominal values.

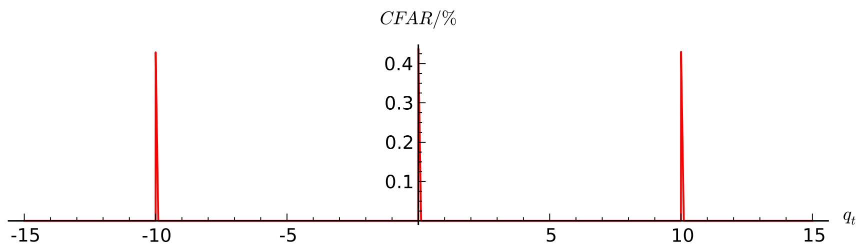

Figure 12. Full guardband CFAR as a function of the measured value.

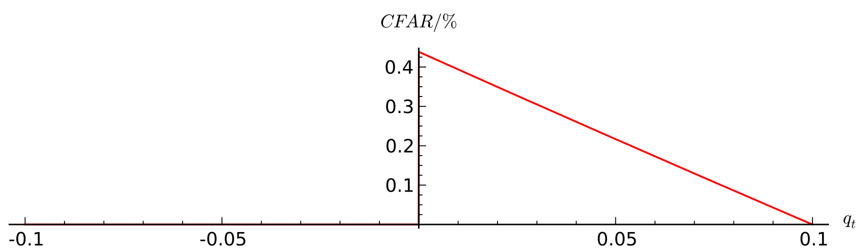

Figure 13. Full guardband CFAR near the nominal values.

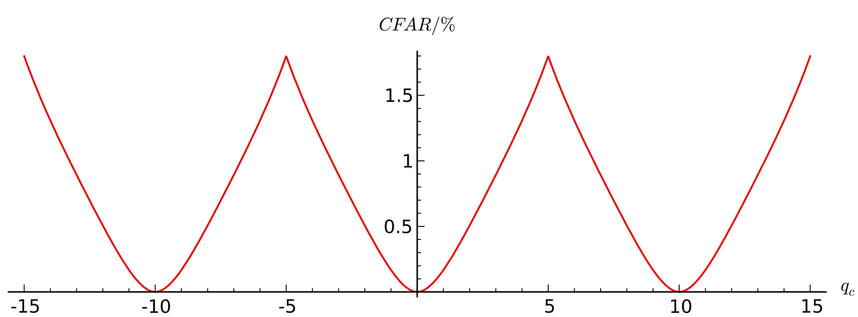

Figure 14. Average full guardband CFAR as a function of the calibration point.

The difference between the measurand and the nominal value creates a residual resolution, similar to that of the sandwich method, that the analysis should consider. Likely though, the narrow acceptance interval more than accounts for that uncertainty. Combining sandwich standards with a full guardband drives risk (due to UUT resolution) to zero throughout the measuring interval.

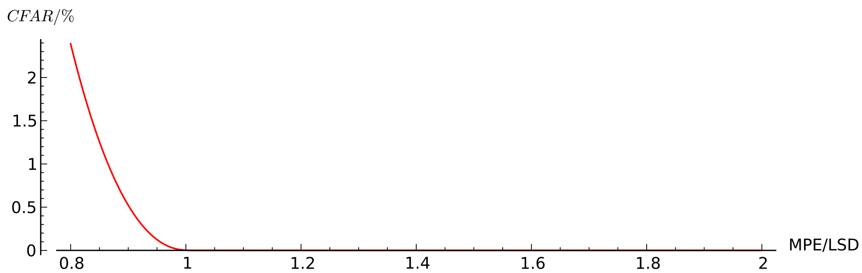

Figure 15. Average full guardband CFAR dependence on MPE.

6.2.4. Change the Spec!

As mentioned in the introduction, specifying MPEs with at least two LSDs greatly lessens the impact. Therefore, if a two-LSD or looser UUT MPE specification suffices for the UUT's intended applications, then simply redefine the specification with the customer and calibrate accordingly. Problem over. Occasionally an instrument comes along from a metrology-challenged manufacturer and its behavior requires an MPE specification term proportional to the measurand but the published specification lacks such a term. Again, redefine the MPE to add the missing term.

6.3. Back to the Drawing Board

Any attempt to reduce uncertainty most logically attacks the largest uncertainty contributors; reducing the smallest contributors yields only diminishing returns. If a given instrument's resolution truly dominates its measurement uncertainty, then the instrument technology controls all other error sources with more capability than indicated. Therefore, why not redesign the instrument display functionality to match, thus improving the resolution and removing the problem? Conversely, if other significant errors exist, why not account for them in the specified MPEs rather than inappropriately claiming a simple $\pm n$ LSD MPE? Likewise, if repeatability dominates, why not address that problem? Defining the measurand as a sufficiently large sample will work if acceptable to the customer and no more practical solutions arise. If the instrument doesn't merit the trouble, we may replace it.

One reason: With a given technology, correcting the largest uncertainty source may cost more than correcting another. In such a case, resolving the issue by expensive redesign or widening the MPE specification will weaken the product's apparent competitiveness in the market. However, we speculate that in the long run, a specification that openly represents the instrument's true performance and facilitates easy verification will attract and retain more loyal customers and profits.

7. A Resolution

In light of recent proposals to ignore resolution and other UUT errors in the calibration uncertainty budget, this paper has explored the ramifications. With a few noted exceptions, we identified an apparent weakness in the extant uncertainty literature regarding the lack of justification for including UUT error sources in instrument calibrations. The prevalent guidance tends toward a notion to automatically include UUT errors. Since methods exist to circumvent UUT errors, metrologists may want to consider those options first. On the other hand, recommendations now exist to exclude UUT errors without mentioning any appropriate methodology to actually eliminate their effects. Metrologists may want to think hard about that.

We have demonstrated that the internal errors determining an instrument's state of calibration reflect the most appropriate measurand for both conformity assessments and providing a calibration correction. Estimating the internal measurands agrees with the philosophy of calibrating instruments like we use them, not as convenient to pass them. The external indication does not suffice as a measurand without specific precautions that actually remove most or all of the UUT error contributions. In summary, the following guidelines and principles have emerged:

- Do not include UUT uncertainties in the budget arbitrarily; know why you do it.
- Do not exclude any uncertainties the calibration process has not thoroughly marginalized.
- Do not round measurement results to the resolution limit if you expect to avoid resolution uncertainty.
- A measurand defined as an uncertainty-free UUT indication does not serve to
  - Adequately calibrate the instrument,
  - Address performance over a measuring interval,
  - Conformance assess an instrument with acceptable risk.
- Many solutions exist to adequately reduce and account for UUT errors.
- Improving the metrology works if you try; ignoring the UUT errors does not.
- All errors belong to the calibration laboratory and flow downstream to the user; reduce them where practical and quantify their remaining uncertainty.

We also recommend that the appropriate parties revise any industry standards that promote calibration via the external measurand without accounting for UUT uncertainties or specifying adequate measurement process details to eliminate the errors in question. Joe Petersen suggested [20] that *ISO 17025*'s sampling section [8, 5.7] might apply to test point selection since the selected points sample the instrument's measuring interval. We would tend to interpret it that way and suggest that the upcoming *17025* revision clarify that point. Under resolution conditions, we should not only properly sample the entire measuring interval, but also the digital quantization.

Metrology seeks metrological compatibility and truth–the degree to which observations agree with theory–to smaller and smaller uncertainties every day. If we attempt to push technological limits, we should expect to pay the commensurate price. If verifying conformance to an instrument's resolution costs more than the benefit realized, we should relax the MPEs; if not, we should pay the price of correct verification.

8. Acknowledgments

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Appendix

This section details and derives the equations and results quoted in the main text. First we list (Table 1) definitions from outside sources and our symbols:

True, internal, displayed, nominal, reference values	q, q_i, q_d, q_n, q_x
Calibration point, test or usage point	q_c, q_t
UUT errors-indication (bias), internal, rounding (display)	e_b, e_i, e_d
Calibration error excluding UUT effects, observed UUT bias	e_x, δ
Calibration uncertainty excluding or including UUT effects	u_x, u_p
Neutral calibration uncertainty	u
UUT external and internal error uncertainties	u_b, u_i
Resolution, resolution uncertainty [15]	$r, u_{\text{res}} = \frac{1}{6}\sqrt{3}r$
Sum of other MPE specification components	s_o
Maximum permissible error [5]	MPE
Test uncertainty ratio [1]	$\text{TUR} = \frac{\text{MPE}}{2\,u}$
RSS-guardbanded acceptance limit [22]	$A_{\text{rss}} = \sqrt{\text{MPE}^2 - 4\,u^2}$
ILAC G8-guardbanded acceptance limit [23]	$A_{g_8} = \text{MPE} - 2\,u$
Lower and upper internal error limits at calibration	l_l, l_u
Lower and upper internal error limits during use	t_l, t_u
UUT's conformance and non-conformance probability	p_c, p_n
Conditional FA risk and mean over the measuring interval	$\text{CFAR}, \mu_{\text{CFAR}}$

Table 1. Definitions and terminology

Resolution Conditions

We define resolution conditions as those under which the UUT resolution approaches or consumes the UUT's MPE. For generality, we begin with a UUT MPE specification that combines n LSDs with other unspecified components,

$$\mathrm{MPE} = nr + s_o. \tag{1}$$

The traditional approach would then compute

$$u_p = \sqrt{u_{\mathrm{res}}^2 + u_x^2}, \tag{2}$$

and substituting Eqs. (1) and (2) into the TUR expression with $u = u_p$ yields

$$\mathrm{TUR} = \frac{nr + s_o}{2\sqrt{\frac{1}{12}r^2 + u_x^2}}. \tag{3}$$

Assuming the MPE specification includes only resolution ($s_o = 0$), $r > 0$, and a perfect measurement process ($u_x = 0$) converts Eq. (3) into

$$\mathrm{TUR} = \sqrt{3}n. \tag{4}$$

The same assumptions likewise transform the RSS- and G8-guardbanded acceptance limit expressions to

$$A_{\mathrm{RSS}} = \frac{1}{3}\sqrt{3\,n^2 - 1}\sqrt{3}r, \tag{5}$$

$$A_{g_8} = \frac{1}{3}\left(\sqrt{3}n - 1\right)\sqrt{3}r. \tag{6}$$

Table 2's numeric results for various n show that, by resolution alone, we fail the typical TUR $\geq$ 4:1 criteria for $n < 3$:

n	TUR	A_{RSS}/r	A_{g_8}/r
3	5.2:1	2.9	2.4
2	3.5:1	1.9	1.4
1	1.7:1	0.82	0.42

Table 2. Resolution conditions

Table 2's one LSD and larger guardbanded acceptance limits, however, might save us in a given measurement scenario, so we define *resolution conditions* as $n < 2$.

Laboratory's Indication MDR

TUR problems boil down to measurement decision risk–FAs and FRs. An FA on the UUT indication means the observed indication error conforms to the MPE, $\delta \leq \text{MPE}$, when the true error does not, $e_b > \text{MPE}$. Figure 1 defines $\delta = e_b - e_x$ and thus this scenario's MDR arises from the measurement error e_x and its uncertainty u_x. References [6], and to a lesser extent, [7], detail MDR so we will not repeat that work here. To concentrate on UUT resolution effects, we take an otherwise perfect measurement with $e_x = 0$ and $u_x = 0$. That leaves $\delta = e_b$, which substituted into the FA conditions gives

$$e_b > e_b. \tag{7}$$

An FR means the observed indication error does not conform to the MPE, $\delta > \text{MPE}$, when the true error conforms, $e_b \leq \text{MPE}$, and the same logic and substitution leads to $e_b > e_b$, the same result.

Equation 7 never holds, so we conclude that UUT resolution never causes an FA or FR on the UUT indication under test. This makes sense since, if we calculated $e_b = q_d - q$ perfectly, then we would have no error or uncertainty to cause MDR because we know q_d perfectly and we know q perfectly if $u_x = 0$.

User's Indication MDR

Most likely however, the UUT user cares more about indication errors at the the UUT usage points than at the lab's certification points. So what CFAR does a calibration ignoring UUT resolution expose the user to?

Consider how well a UUT indication bounds its associated e_i: The certification accepts $e_b \leq \text{MPE}$, which includes any q_i that rounds to a conforming indication, in other words, out to $\frac{1}{2}r$ past the worst permissible indications. For a true value perfectly equal to some nominal UUT indication, q_n, any internal error

$$|e_i| \leq \text{MPE} + \frac{1}{2}r \tag{8}$$

produces a conforming indication.[16] A realistic true value, however, will fall somewhere between indications so we take q_n as the indication just below q. The minimum accepted internal error then loses the extra LSD it now crosses and gains the deviation $\Delta_l = q - q_n$; likewise, the maximum accepted internal error loses one LSD and gains the deviation $\Delta_u = r - \Delta_l$. Adjusting Eq. 8 by $r - \Delta_l$ and $r + (\Delta_l - r)$, respectively, yields e_i's lower and upper bounds,

$$l_l = \left(-\text{MPE} - \frac{1}{2}r\right) + (r - \Delta_l),$$

[16]We use the symbols $\leq, \geq$ but hardware behavior and noise at the rounding boundaries make the choice between open and closed intervals immaterial.

$$l_u = \left(\text{MPE} + \frac{1}{2}r\right) + (-\Delta_l), \tag{9}$$

for an accepted e_b. To merge Eqs. 8 and 9 and eliminate q_n, we use modulo arithmetic, note that $\text{modulo}\,(q_n, r) = 0$, so $\text{modulo}\,(q - q_n, r) = \text{modulo}\,(q, r)$, and write

$$\begin{aligned} l_l &= -\text{MPE} - \frac{1}{2}r + \text{modulo}\,(r - q, r)\,, \\ l_u &= \text{MPE} + \frac{1}{2}r - \text{modulo}\,(q, r)\,. \end{aligned} \tag{10}$$

Note that the internal error bounds depend sensitively on the test point location, something that does not occur in normal calibrations. Given e_i's pdf[17], $f_{\text{int}}\,(e_i)$, the probability that the UUT indication's error at q exceeds its MPE comes to simply one minus the integral over the interval [6] between the Eq. 10 limits,

$$p_n = -\int_{l_l}^{l_u} f_{\text{int}}\,(e_i)\; de_i + 1. \tag{11}$$

If we assume that many physical and circuit factors contribute to the internal error such that a normal distribution approximates $f_{\text{int}}\,(e_i)$, then Eq. 11 becomes

$$p_n = \frac{1}{2}\,\text{erf}\left(\frac{\sqrt{2}l_l}{2\,u_i}\right) - \frac{1}{2}\,\text{erf}\left(\frac{\sqrt{2}l_u}{2\,u_i}\right) + 1. \tag{12}$$

Given then, that we accept the UUT indication at a calibration point q_c, what CFAR do we impose on the user at an arbitrary usage point q_t? CFAR arises anytime the internal error acceptance limits at q_c exceed those at q_t. Therefore, we integrate $f_{\text{int}}\,(e_i)$ from q_t's acceptance limits, t_l and t_u, out to q_c's and condition the result [6] on the conformance probability $p_c = -p_n + 1$, taking zero for calibration bounds tighter than usage bounds:

$$\text{CFAR} = \frac{\max\left(0, \int_{l_l}^{t_l} f_{\text{int}}\,(e_i)\; de_i\right) + \max\left(0, \int_{t_u}^{l_u} f_{\text{int}}\,(e_i)\; de_i\right)}{p_c}. \tag{13}$$

For a normal $f_{\text{int}}\,(e_i)$ we have

$$\text{CFAR} = -\frac{2\left(\max\left(0, \frac{1}{2}\,\text{erf}\left(\frac{\sqrt{2}l_u}{2\,u_i}\right) - \frac{1}{2}\,\text{erf}\left(\frac{\sqrt{2}t_u}{2\,u_i}\right)\right) + \max\left(0, -\frac{1}{2}\,\text{erf}\left(\frac{\sqrt{2}l_l}{2\,u_i}\right) + \frac{1}{2}\,\text{erf}\left(\frac{\sqrt{2}t_l}{2\,u_i}\right)\right)\right)}{\text{erf}\left(\frac{\sqrt{2}l_l}{2\,u_i}\right) - \text{erf}\left(\frac{\sqrt{2}l_u}{2\,u_i}\right)}. \tag{14}$$

[17]Probability density function

As a calibration quality metric, we calculate the user's expected CFAR by averaging Eq. 13 over q_t's interval:

$$\mu_{\text{CFAR}} = \frac{\int_0^r \text{CFAR}\left(q_t\right)\, dq_t}{r}. \tag{15}$$

For multiple calibration points, each q_{ci} contributes acceptance limits L_i and U_i according to Eq. 10, the tightest of which become the overall acceptance limits

$$\begin{aligned} l_l &= \max\left(L_1, L_n\right), \\ l_u &= \min\left(U_1, U_n\right), \end{aligned} \tag{16}$$

which then apply in the various CFAR equations. Since the modulo function produces acceptance limits that repeat over every resolution interval, further calibration points over the measuring interval yield no further benefit unless their point position within the resolution interval varies. Thus, we may calibrate the UUT at multiple nominal indications and reduce CFAR not a bit, or vary the positioning and reduce UUT resolution effects accordingly.

Slewing

We may avoid the UUT resolution and directly estimate e_i by slewing q to find a point at which q_d's LSD increases one count. This occurs when $q_i = q_n + \frac{1}{2}\, r$. Substituting $q_i = q + e_i$ and the approximation $q = q_x$, and solving for the internal error yields the estimate

$$e_i = q_n - q_{x_1} + \frac{1}{2}\, r. \tag{17}$$

Likewise analyzing the point at which the LSD decreases one count, $q_i = q_n - \frac{1}{2}\, r$, and averaging the results improves the estimate to

$$e_i = q_n - \frac{1}{2}\, q_{x_1} - \frac{1}{2}\, q_{x_2}. \tag{18}$$

Implicit Guardbands

We now determine the conditions under which various guardbanding methods produce acceptance limits no tighter than the effective MPE, so that the UUT rounding implicitly guardbands the measurement result. This requires an MPE specification that includes some minimum component s_o added to the LSD component. Note: Though the RSS guardband offers a nicely balanced PFA[18] and PFR[19] [22], none of the guardbanding methods covered here deliver a selectable PFA or PFR. We refer the reader to *RP-18* [6] for guardbands that achieve a desired risk target.

[18] Probability of false accept

[19] Probability of false reject

RSS Guardband

The RSS guardband will most likely work implicitly. Combining the Table 1 RSS guardband definition with Eqs. (1) and (2) gives the RSS acceptance limits

$$A_{\text{rss}} = \sqrt{(nr + s_o)^2 - \frac{1}{3}\, r^2 - 4\, u_x^2}. \tag{19}$$

Solving for s_o with the condition $A \geq nr$ produces the RSS implicit guardband condition

$$s_o \geq -nr + \frac{1}{3}\, \sqrt{(3\, n^2 + 1) r^2 + 12\, u_x^2}\sqrt{3}. \tag{20}$$

80 % Guardband

This simple guardband takes 80 % of the MPE as the acceptance limit, or $A_{80} = \frac{4}{5}\,\text{MPE}$. Substituting Eq. (1) and requiring $A \geq nr$ yields

$$s_o \geq \frac{1}{4}\, nr. \tag{21}$$

ILAC G8 Guardband

The aggressive *ILAC G8* guardband with its high PFR penalty will least likely produce an implicit guardband. From Table 1 and Eqs. (1) and (2) we obtain the G8 acceptance limits

$$A_{g8} = nr + s_o - 2\, \sqrt{\frac{1}{12}\, r^2 + u_x^2}. \tag{22}$$

Requiring $A \geq nr$ leads to the *ILAC G8* implicit guardband condition

$$s_o \geq 2\, \sqrt{\frac{1}{12}\, r^2 + u_x^2}. \tag{23}$$

Full Guardband

Let us define the nominal UUT indication, q_n, as the measurand, q, rounded to the nearest LSD, or $q_n = r\left\lfloor \frac{q}{r} + \frac{1}{2} \right\rfloor$, and require the UUT to indicate that exactly before releasing it. We call this a full guardband. The UUT will meet that criterion, rounding its internal value, q_i, to q_n if $-\frac{1}{2}\, r \leq q_i - q_n$ and $q_i - q_n < \frac{1}{2}\, r$. Substituting $q_i = q + e_i$ requires $-\frac{1}{2}\, r \leq e_i + q - q_n$ and $e_i + q - q_n < \frac{1}{2}\, r$. Using the nominal value definition and noting that $\text{modulo}\,(a, b) = a - b\left\lfloor \frac{a}{b} \right\rfloor$ such that $\text{modulo}\left(q + \frac{1}{2}\, r, r\right) = -r\left\lfloor \frac{2\,q + r}{2\,r} \right\rfloor + q + \frac{1}{2}\, r$, we find that the full guardband bounds the internal error with limits

$$\begin{aligned} l_l &= -\text{modulo}\left(q + \frac{1}{2}\, r, r\right), \\ l_u &= r - \text{modulo}\left(q + \frac{1}{2}\, r, r\right). \end{aligned} \tag{24}$$